The Chevalley group $G_2(2)$ of order 12096 and the octonionic root system of E_7

Mehmet Koca*
Department of Physics, College of Science, Sultan Qaboos University,
PO Box 36, Al-Khod 123, Muscat, Sultanate of Oman

Ramazan Koç†
Department of Physics, Faculty of Engineering University of Gaziantep, 27310 Gaziantep, Turkey

Nazife Ö. Koca‡
Higher College of Technology, Al-Khuwair, Muscat, Sultanate of Oman
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The octonionic root system of the exceptional Lie algebra E_8 has been constructed from the quaternionic roots of F_4 using the Cayley-Dickson doubling procedure where the roots of E_7 correspond to the imaginary octonions. It is proven that the automorphism group of the octonionic root system of E_7 is the adjoint Chevalley group $G_2(2)$ of order 12096. One of the four maximal subgroups of $G_2(2)$ of order 192 preserves the quaternion subalgebra of the E_7 root system. The other three maximal subgroups of orders 432, 192 and 336 are the automorphism groups of the root systems of the maximal Lie algebras $E_6 \times U(1)$, $SU(2) \times SO(12)$ and $SU(8)$ respectively. The 7-dimensional manifolds built with the use of these discrete groups could be of potential interest for the compactification of the M-theory in 11-dimension.

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INTRODUCTION

The Chevalley groups are the automorphism groups of the Lie algebras defined over the finite fields [1]. The group $G_2(2)$ is the automorphism group of the Lie algebra g_2 defined over the finite field F_2 which is one of the finite subgroups of the Lie group G_2 [2]. Here we prove that it is the automorphism group of the octonionic root system of the exceptional Lie group E_7 .

The exceptional Lie groups are fascinating symmetries arising as groups of invariants of many physical models suggested for fundamental interactions. In the sequel of grand unified theories(GUT's) after $SU(5) \approx E_4$ [3], $SO(10) \approx E_5$ [4] the exceptional group E_6 [5] has been suggested as the largest GUT for a single family of quarks and leptons. The 11-dimensional supergravity theory admits an invariance of the non-compact version of $E_7[E_{7(-7)}]$ with a compact subgroup $SU(8)$ as a global symmetry [6]. The largest exceptional group E_8, originally proposed as a grand unified theory [7] allowing a three family interaction of E_6, has naturally appeared in the heterotic string theory as the $E_8 \times E_8$ gauge symmetry [8].

The infinite tower of the spin representations of $SO(9)$, the little group of the 11-dimensional M-theory, seems to be unified in the representations of the exceptional group F_4 [9]. Moreover, it has been recently shown that the root system of F_4 can be represented with discrete quaternions whose automorphism group is the direct product of two binary octahedral groups of order $48 \times 48 = 2304$ [10].

The smallest exceptional group G_2, the automorphism group of octonion algebra, turned out to be the best candidate as a holonomy group of the 7-dimensional manifold for the compactification of M-theory [11]. For a " topological M-theory" [12] one may need a crystallographic structure in 7-dimensions. In this context the root lattices of the Lie algebras of rank-7 may play some role, such as those of $SU(8)$, E_7 and the other root lattices of rank-7 Lie algebras. The $SU(8)$ is a maximal subgroup of E_7 therefore it is tempting to study the E_7 root lattice. Here a miraculous happens! The root system of E_7 can be described by the imaginary discrete octonions [13]. The Weyl group $W(E_7)$ is isomorphic to a finite subgroup of $O(7)$ which is the direct product $Z_2 \times SO_7(2)$ where the latter group is the adjoint Chevalley group of order $2^9.3^4.5.7$ [14]. However, the Weyl group $W(E_7)$ does not preserve the octonion algebra. When one imposes the invariance of the octonion algebra on the transformations of the E_7 roots one obtains a finite subgroup of G_2 , as expected, the adjoint Chevalley group $G_2(2)$ of order 12096 [13, 15]. A G_2 holonomy group of the 7- dimensional manifold admitting the discrete symmetry $G_2(2)$ may turn out to be useful for $E_{7(-7)}$ is related to the 11- dimensional supergravity theory.

In what follows we discuss the mathematical structure of the adjoint Chevalley group $G_2(2)$ using the126 non-zero octonionic roots of E_7 without referring to its matrix representation [16].

In section 2 we construct the octonionic roots of E_8 [13, 17] using the two sets of quaternionic roots of F_4 which

	$SU(3)$	$SP(3)$	F_4
$SU(3)$	$SU(3)\times SU(3)$	$SU(6)$	E_6
$SP(3)$	$SU(6)$	$SO(12)$	E_7
F_4	E_6	E_7	E_8

TABLE I: Magic Square

follows the magic square structure [18] where imaginary octonions represent the roots of E_7 . First we build up a maximal subgroup of $G_2(2)$ of order 192 which preserves the quaternionic decomposition of the octonionic roots of E_7 . It is a finite subgroup of $SO(4)$ .Section 3 is devoted to a discussion on the embeddings of the group of order 192 in the $G_2(2)$. In section 4 we study the maximal subgroups of $G_2(2)$ and their relevance to the root systems of the maximal Lie algebras of E_7. Finally, in section 5, we discuss the use of our method in physical applications and elaborate the various geometrical structures.

OCTONIONIC ROOT SYSTEM OF E_8

In the reference [13] we have shown that the octonionic root system of E_8 can be constructed by doubling two sets of quaternionic root system of F_4 [10] via Cayley-Dickson procedure. Symbolically we can write,

$$(F_4, F_4) = E_8 \tag{1}$$

where the short roots of F_4 match with the short roots of the second set of F_4 roots and the long roots match with the zero roots. Actually (1) follows from the magic square given by Table 1. The quaternionic scaled roots of F_4 can be given as follows:

$$F_4 : T \oplus \frac{T'}{\sqrt{2}} \tag{2}$$

where $T \oplus T'$ are the set of elements of the binary octahedral group, compactly written as

$$\begin{aligned} T &= V_0 \oplus V_+ \oplus V_- \\ T\prime &= V_1 \oplus V_2 \oplus V_3. \end{aligned} \tag{3}$$

More explicitly, the set of quaternions $V_0, V_+, V_-, V_1, V_2, V_3$ read

$$\begin{aligned} V_0 &= \{\pm 1, \pm e_1, \pm e_2, \pm e_3\} \\ V_+ &= \left\{\frac{1}{2}\pm 1 \pm e_1 \pm e_2 \pm e_3\right\}, \text{even} \quad \text{number} \quad \text{of} \quad (+)\text{signs} \\ V_- &= \overline{V_+} = \left\{\frac{1}{2}\pm 1 \pm e_1 \pm e_2 \pm e_3\right\}, \text{even} \quad \text{number} \quad \text{of} \quad (-)\text{signs} \end{aligned} \tag{4}$$

($\overline{V_+}$ is the quaternionic conjugate of V_+)

$$\begin{aligned} V_1 &= \left\{\frac{1}{\sqrt{2}}(\pm 1 \pm e_1), \frac{1}{\sqrt{2}}(\pm e_2 \pm e_3)\right\} \\ V_2 &= \left\{\frac{1}{\sqrt{2}}(\pm 1 \pm e_2), \frac{1}{\sqrt{2}}(\pm e_3 \pm e_1)\right\} \\ V_3 &= \left\{\frac{1}{\sqrt{2}}(\pm 1 \pm e_3), \frac{1}{\sqrt{2}}(\pm e_1 \pm e_2)\right\} \end{aligned} \tag{5}$$

where $e_i (i = 1, 2, 3)$ are the imaginary quaternionic units.

Here T is the set of quaternionic elements of the binary tetrahedral group which represents the root system of $SO(8)$ and $\frac{T'}{\sqrt{2}}$ represents the weights of the three 8-dimensional representations of $SO(8)$ or, equivalently, T and $\frac{T'}{\sqrt{2}}$ represent the long and short roots of F_4 respectively. The geometrical meaning of these vectors are also interesting [19]. Here each of the sets V_0, V_+, V_- represent a hyperoctahedron in 4-dimensional Euclidean space. The set T is

also known as a polytope $\{3,4,3\}$ called 24-cell [20]. Its dual polytope is $T\prime$ where $V_i(i=1,2,3)$ are the duals of the octahedron in T. Any two of the sets V_0, V_+, V_- form a hypercube in 4-dimensions. Using the Cayley-Dickson doubling procedure one can construct the octonionic roots of E_8 as follows:

$$\begin{aligned}
(T,0) &= T,(0,T)=e_7T \\
(\frac{V_1}{\sqrt{2}},\frac{V_1}{\sqrt{2}}) &= \frac{1}{\sqrt{2}}(V_1+e_7V_1) \\
(\frac{V_2}{\sqrt{2}},\frac{V_3}{\sqrt{2}}) &= \frac{1}{\sqrt{2}}(V_2+e_7V_3) \\
(\frac{V_3}{\sqrt{2}},\frac{V_2}{\sqrt{2}}) &= \frac{1}{\sqrt{2}}(V_3+e_7V_2)
\end{aligned} \tag{6}$$

where e_1 , e_2 and e_7 are the basic imaginary units to construct the other units of octonions $1, e_1, e_2, e_3 = e_1e_2, e_4 = e_7e_1, e_5 = e_7e_2, e_6 = e_7e_3$. They satisfy the algebra

$$e_ie_j = -\delta_{ij} + \phi_{ijk}e_k, \quad (i,j,k=1,2,...,7)$$

where ϕ_{ijk} is totally anti-symmetric under the interchange of the indices i,j,k and take the values $+1$ for the indices $123, 246, 435, 367, 651, 572, 741$ [21]. The set of E_8 roots in (6) can also be compactly written as the sets of octonions

$$\pm \quad 1, \frac{1}{2}(\pm1 \pm e_a \pm e_b \pm e_c), \tag{7}$$

$$\pm \quad e_i(i=1,2,...,7), \frac{1}{2}(\pm e_d \pm e_f \pm e_g \pm e_h) \tag{8}$$

where the indices take $abc = 123, 156, 147, 245, 267, 346, 357$ and $dfgh$ =1246, 1257, 1345, 1367, 2356, 2347, 4567. When ±1 represent the non-zero roots of $SU(2)$ the imaginary roots in (8)which are orthogonal to ±1 represent the roots of E_7. The decomposition of the roots in(7-8) represents the branching of E_8 under its maximal subalgebra $SU(2)\times E_7$ where the 112 roots in (7) are the weights $(\underline{2},\underline{56})$.

A subset of roots of F_4 consisting of imaginary quaternions constitute the roots of subalgebra $SP(3)$ with the short and long roots represented by

$$\begin{aligned}
&SP(3) \ : \\
&\text{long} \quad \text{roots} \ : \ V_0' = \{\pm e_1, \pm e_2, \pm e_3\}\,; \\
&\text{short} \quad \text{roots} \ : \ \frac{V_1'}{\sqrt{2}} = \left\{\frac{1}{2}(\pm e_2, \pm e_3)\right\}, \frac{V_2'}{\sqrt{2}} = \left\{\frac{1}{2}(\pm e_3, \pm e_1)\right\}, \frac{V_3'}{\sqrt{2}} = \left\{\frac{1}{2}(\pm e_1, \pm e_2)\right\}
\end{aligned} \tag{9}$$

From the magic square one can also write the roots of E_7 in the form $(SP(3), F_4)$ consisting of only imaginary octonions which can further be put in the form

$$\begin{aligned}
(V_0',0) &= V_0',(0,T)=e_7T \\
(\frac{V_1'}{\sqrt{2}},\frac{V_1}{\sqrt{2}}) &= \frac{1}{\sqrt{2}}(V_1'+e_7V_1) \\
(\frac{V_2'}{\sqrt{2}},\frac{V_3}{\sqrt{2}}) &= \frac{1}{\sqrt{2}}(V_2'+e_7V_3) \\
(\frac{V_3'}{\sqrt{2}},\frac{V_2}{\sqrt{2}}) &= \frac{1}{\sqrt{2}}(V_3'+e_7V_2)
\end{aligned} \tag{10}$$

The roots in (10) also follows from a Coxeter-Dynkin diagram of E_8 where the simple roots represented by octonions depicted in Figure 1. As we stated in the introduction, the automorphism group of octonionic root system of E_7 is the adjoint Chevalley group $G_2(2)$, a maximal subgroup of the Chevalley group $SO_7(2)$. Below we give a proof of this assertion and show how one can construct the explicit elements of $G_2(2)$ without any reference to a computer calculation of the matrix representation.

We start with a theorem [22] which states that the automorphism of octonions that take the quaternions H to itself form a group $[p,q]$, isomorphic to $SO(4) \approx \frac{SU(2)\times SU(2)}{Z_2}$. Here p and q are unit quaternions. In a different work [23]

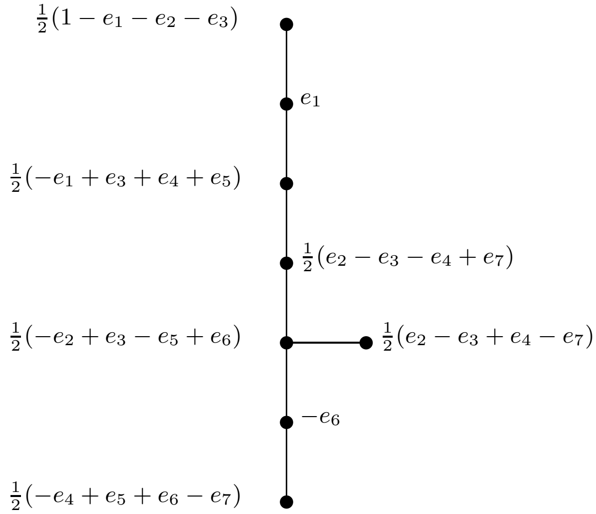

FIG. 1: The Coxeter-Dynkin diagram of E_8 with quaternionic simple roots

we have studied some finite subgroups of $O(4)$ generated by the transformations

$$\begin{aligned} [p,q] &: r \rightarrow prq \\ [p,q]^* &: r \rightarrow p\bar{r}q \end{aligned} \tag{11}$$

where $[p,q]$ represents an $SO(4)$ transformation preserving the norm $r\bar{r}=\bar{r}r$ of the quaternion r. More explicitly, it has been shown in [22] that the group element $[p,q]$ acts on the Cayley-Dickson double quaternion as

$$[p,q] : H + e_7H \rightarrow pH\bar{p} + e_7pHq \tag{12}$$

Now we use this theorem to prove that the transformations on the root system of E_7 in (10) preserving the quaternion subalgebra form a finite subgroup of $SO(4)$ of order 192. In reference [10] we have shown that the maximal finite subgroup of $SO(3)$ which preserves the set of quaternions $V\prime_0 = \{\pm e_1, \pm e_2, \pm e_3\}$ representing the long roots of $SP(3)$ as well as the vertices of an octahedron is the octahedral group written in the form $[t,\bar{t}] \oplus [t',\bar{t}]$ where $t \in T$ and $t' \in T'$. On the other hand e_7T is left invariant under the transformations $[p,q] \oplus [p',q']$, $(p,q \in T; p',q' \in T\prime)$. Therefore the largest group preserving the structure $(V_0',0) = V_0'$, $(0,T) = e_7T$ is a finite subgroup of $SO(4)$ of order 576. We will see that actually we look for a subgroup of this group because it should also preserve the set of roots

$$\frac{1}{\sqrt{2}}(V_1'+e_7V_1), \frac{1}{\sqrt{2}}(V_2'+e_7V_3), \frac{1}{\sqrt{2}}(V_3'+e_7V_2) \tag{13}$$

as well as keeping the form of (12) invariant.

A multiplication table shown in Table 2 for the elements of the binary octahedral group [19] will be useful to follow the further discussions. Equation(12) states that the transformation $pH\bar{p}$ fixes the scalar part of the quaternion H. Therefore the transformation in (12) acting on the root system of E_8 in (6) will yield the same result when (12) acts on the roots of E_7 in (10). Now we check the transformation (12) acting on the roots in (13) and seek the form of $[p,q]$ which preserves (13). More explicitly , we look for the invariance

$$\begin{aligned} &\frac{1}{\sqrt{2}}(pV_1'\bar{p}+e_7pV_1q) \oplus \frac{1}{\sqrt{2}}(pV_2'\bar{p}+e_7pV_3q) \oplus \frac{1}{\sqrt{2}}(pV'\bar{p}_3+e_7pV_2q) \\ = &\frac{1}{\sqrt{2}}(V_1'+e_7V_1) \oplus \frac{1}{\sqrt{2}}(V_2'+e_7V_3) \oplus \frac{1}{\sqrt{2}}(V_3'+e_7V_2). \end{aligned} \tag{14}$$

We should check all pairs in $[V_0 \oplus V_+ \oplus V_-$, $V_0 \oplus V_+ \oplus V_-$] and see that only the set of elements $[V_0$, $V_0]$, $[V_+$, $V_0]$, $[V_-, V_0]$ satisfy the relation (14). Just to see why $[V_+, V_+]$, for example, does not work let us apply it on the set of

	V_0	V_+	V_-	V_1	V_2	V_3
V_0	V_0	V_+	V_-	V_1	V_2	V_3
V_+	V_+	V_-	V_0	V_3	V_1	V_2
V_-	V_-	V_0	V_+	V_2	V_3	V_1
V_1	V_1	V_2	V_3	V_0	V_+	V_-
V_2	V_2	V_3	V_1	V_-	V_0	V_+
V_3	V_3	V_1	V_2	V_+	V_-	V_0

TABLE II: Multiplication table of the binary octahedral group

roots $\frac{1}{\sqrt{2}}(V_1' + e_7 V_1)$:

$$[V_+, V_+] : \frac{1}{\sqrt{2}}(V_1' + e_7 V_1) \rightarrow \frac{1}{\sqrt{2}}(V_+ V_1' V_- + e_7 V_+ V_1 V_+).$$

Using Table 2 we obtain that

$$\frac{1}{\sqrt{2}}(V_1' + e_7 V_1) \rightarrow \frac{1}{\sqrt{2}}(V_2' + e_7 V_1)$$

which does not belong to the set of roots of E_7. Similar considerations eliminate all the subsets of elements in $[T, T]$ but leaves only $[T, V_0]$. Note that $[V_+, V_0]^3 = [V_0, V_0]$ and it permutes the three sets of roots of E_7 in (13). Now we study the action of $[T', T']$ on the roots in (13). We can easily prove that the set of elements $[V_1, V_1]$ does the job:

$$\begin{aligned} [V_1, V_1] \ : \ \frac{1}{\sqrt{2}}(V_1' + e_7 V_1) &\rightarrow \frac{1}{\sqrt{2}}(V_1' + e_7 V_1) \\ \frac{1}{\sqrt{2}}(V_2' + e_7 V_3) &\leftrightarrow \frac{1}{\sqrt{2}}(V_3' + e_7 V_2) \end{aligned} \tag{15}$$

We can check easily that the set of elements $[V_2, V_1]$ and $[V_3, V_1]$ also satisfy the requirements. Note that $[V_i, V_1]^2 = [V_0, V_0]$, $(i = 1, 2, 3)$; any one of these set of elements, while preserving one set of roots in(13), exchange the other two.

We conclude that the subset of elements of the group $[p, q] \oplus [p', q']$, $(p, q \in T; p', q' \in T')$ which preserve the root system of E_7 is the group of elements $[T, V_0] \oplus [T', V_1]$ of order 192 with 17 conjugacy classes. It is interesting to note that $[V_0, V_0]$ is an invariant subgroup of order 32 of the group $[T, V_0] \oplus [T', V_1]$. Actually it is the direct product of the quaternion group with itself consisting of elements $V_0 = \{\pm 1, \pm e_1, \pm e_2, \pm e_3\}$. The set of elements $[T, V_0] \oplus [T', V_1]$ now can be written as the union of cosets of $[V_0, V_0]$ where the coset representatives can be obtained from, say, $[V_+, V_0]$ and $[V_1, V_1]$. When $[V_0, V_0]$ is taken as a unit element then $[V_+, V_0]$ and $[V_1, V_1]$ generate a group isomorphic to the symmetric group S_3. Symbolically, the group of interest can be written as the semi-direct product of the group $[V_0, V_0]$ with S_3 which is a maximal subgroup of order 576 of the direct product of two binary octahedral group.

It is also interesting to note that the group $[T, T] \oplus [T', T']$ has another maximal subgroup of order 192 with 13 conjugacy classes whose elements can be written as

$$[V_0, V_0] \oplus [V_+, V_-] \oplus [V_-, V_+] \oplus [V_1, V_1] \oplus [V_2, V_2] \oplus [V_3, V_3] \tag{16}$$

This group does not preserve the root system of E_7 ,however, it preserves the quaternion algebra in the set of imaginary octonions $\pm e_i (i = 1, 2, ..., 7)$. This is also an interesting group which turns out to be maximal in an another finite subgroup of $G_2(2)$ of order 1344 [24]. The group in (16) can also be written as the semi-direct product of $[V_0, V_0]$ and S_3, however, two groups are not isomorphic because the symmetric group S_3 here is generated by $[V_+, V_-]$ and $[V_1, V_1]$ instead of $[V_+, V_0]$ and $[V_1, V_1]$ as in the previous case.

63 EMBEDDINGS OF THE QUATERNION PRESERVING GROUP IN THE CHEVALLEY GROUP

We go back to the equation (6) and note that the binary tedrahedral group $T = V_0 + V_+ + V_-$ played an important role in the above analysis for it represents the root system of $SO(8)$. Any one element of the quaternionic elements

of the hypercube $V_+ + V_- = \frac{1}{2}\{\pm 1 \pm e_1 \pm e_2 \pm e_3\}$ satisfies the relation $p^3 = \pm 1$. Actually we have 112 octonionic elements of this type in the roots of E_8.

We have proven in an earlier paper [23][28] that the transformation

$$b \to ab\bar{a} \tag{17}$$

where $a^3 = \pm 1$ is an associative product of octonions which preserve the octonion algebra. More explicitly, when $e_i (i = 1, 2, ..., 7)$ represent the imaginay octonions the transformation

$$e_i' = ae_i\bar{a}, (a^3 = \pm 1) \tag{18}$$

preserves the octonion algebra

$$e_i'e_j' = (e_ie_j)' = a(e_ie_j)\bar{a}. \tag{19}$$

To work with octonionionic root systems makes life difficult because of nonassociativity. However, the following theorem [25] proves to be useful. Let p be any root of those 112 roots and q be any root of E_8. Consider the transformations on q :

$$\pm p : q_1 \equiv q, q_2 \equiv (p)q(\bar{p}), q_3 \equiv (\bar{p})q(p).$$

It was proven in [25] that q_1, q_2, q_3 form an associative triad $(q_1q_2)q_3 = q_1(q_2q_3)$ satisfying the relations

$$\begin{aligned} &q_1 p \quad \text{for} \quad q_i.\bar{p} = 0, (42 \quad \text{triads}) \\ &q_1q_2q_3 = \begin{cases} -1 \ \text{for} \ q_i.\overline{p} = -1/2 \ \ 18 \quad \text{triads} \\ \ \ 1 \ \ \text{for} \ \ q_i.\overline{p} = 1/2 \ \ \ 18 \quad \text{triads} \end{cases} \end{aligned} \tag{20}$$

Actually this decomposition of E_8 roots is the same as its branching under $SU(2) \times E_7$ where the non-zero roots decompose as $240 = 126 + 2 + (2, 56)$. The first 42 triads are the 126 non-zero roots of E_7 and $\pm\bar{p}$ are those of $SU(2)$. The remaining $36 \times 3 = 108$ roots with $\pm 1, \pm p$ constitute the 112 roots of the coset space. In general one can show that 24triads, out of 42 triads, corresponding to the roots of E_6 are imaginary octonions and the remaining 18 triads are those with non-zero scalar parts. The 9 triads of those octonionic roots which satisfy the relation $q_i.\bar{p} = -\frac{1}{2}$ are imaginary octonions and their negatives satisfy the relation $q_i.\bar{p} = \frac{1}{2}$. When ± 1 represent the roots of $SU(2)$ then all the roots of E_7 are pure imaginary as depicted in Figure 1. For a given octonion p with non-zero real part one can classify the imaginary roots of E_7 as follows:

(i) 72 imaginary octonions which are grouped in 24 triads satisfying the relation $q_i.\bar{p} = 0$

(ii) 27 imaginary roots classified in 9 associative triads whose products satisfy the relation $q_1q_2q_3 = -1$ are the quaternionic units. They represent the weights of the 27 dimensional representation of E_6.

(iii)The remaining 9 triads are the conjugates of those in (ii) and represent the weights of the representation $\overline{27}$ of E_6 .

In the next section we will prove that the root system of E_8 in (6) and equivalently those of E_7 in (10) can be constructed 63 different ways preserving the octonion algebra so that the automorphism group of the octonionic root system of E_7 is the group $G_2(2)$ of order $192 \times 63 = 12096$.

We recall that we have 18 associative triads with non-zero scalar part, each being orthogonal to $\bar{p}$. To distinguish the imaginary octonions for which we keep the notation q_i we denote the roots with non-zero scalar part by r_i satisfying the relation $r_i.\bar{p} = 0$ where $r_i^3 = \pm 1$, $(i = 1, 2, 3)$. They are permuted as follows :

$$r_1, r_2 = pr_1\bar{p}, r_3 = \bar{p}r_1p.$$

The scalar product $r_i.\bar{p} = 0$ can be written as

$$r_ip + \bar{p}\bar{r}_i = \bar{r}_ip + \bar{p}r_i = 0. \tag{21}$$

We can use (20) to show that $r_1r_2 = r_2r_3 = r_3r_1 = p$ with conjugates $\bar{r}_2\bar{r}_1 = \bar{r}_3\bar{r}_2 = \bar{r}_1\bar{r}_2 = \bar{p}$.One can easily show that the octonions r_1, r_2 and r_3 are mutually orthogonal to each other:

$$r_1.r_2 = r_2.r_3 = r_3.r_1 = 0 \to r_1\bar{r}_2 + r_2\bar{r}_1 = r_2\bar{r}_3 + r_3\bar{r}_2 = r_3\bar{r}_1 + r_1\bar{r}_3 = 0 \tag{22}$$

which also implies that $r_1\bar{r}_2, r_2\bar{r}_3, r_3\bar{r}_1$ are imaginary octonions.

The orthogonality of r_1, r_2 and r_3 can be proven as follows. Consider the scalar product

$$r_1.r_2 = \frac{1}{2}[\bar{r}_1(pr_2\bar{p}) + (p\bar{r}_1\bar{p})r_1]. \tag{23}$$

Let us assume without loss of generality that $\bar{p} = 1 - p$, $\bar{r}_1 = 1 - r_1$. Substituting $\bar{p} = 1 - p$ and $\bar{r}_1 = 1 - r_1$ in (23) and using (21) as well as the Moufang identities [22]

$$\begin{aligned} (pq)(rp) &= p(qr)p && (24)\\ p(qrq) &= [(pq)r]q && (25)\\ (qrq)p &= q[r(qp)] && (26) \end{aligned}$$

one can show that $r_1.r_2 = 0$. Similar considerations for the other octonions will prove that the four octonions r_1, r_2, r_3 and $\bar{p}$ are mutually orthogonal to each other so that $\pm r_1, \pm r_2, \pm r_3$ and $\pm\bar{p}$ form the vertices of a hyperoctahedron. Similarly their conjugates forming an orthogonal quarted with their negatives represent the vertices of another hyperoctahedron. The imaginary octonions $r_1\bar{r}_2, r_2\bar{r}_3, r_3\bar{r}_1$ are cyclically rotated to each other in the manner $p(r_1\bar{r}_2)\bar{p} = r_2\bar{r}_3$ (cyclic permutations of $1, 2, 3$) and satisfying the relation $(r_1\bar{r}_2).\bar{p} = \frac{1}{2}$ where the conjugate $r_2\bar{r}_1$ satisfies the relation $(r_2\bar{r}_1).\bar{p} = -\frac{1}{2}$. If we denote by the imaginary octonions $E_1 = r_3\bar{r}_2, E_2 = r_1\bar{r}_3$ and $E_3 = r_2\bar{r}_1$. It is easy to prove the following identities:

$$\begin{aligned} \bar{p} &= \frac{1}{2}(1 - E_1 - E_2 - E_3)\\ r_1 &= \frac{1}{2}(1 + E_1 + E_2 - E_3)\\ r_2 &= \frac{1}{2}(1 - E_1 + E_2 + E_3)\\ r_3 &= \frac{1}{2}(1 + E_1 - E_2 + E_3) \end{aligned} \tag{27}$$

Therefore the set of 24 octonions

$$\left\{\pm 1, \pm E_1, \pm E_2, \pm E_3, \frac{1}{2}\left(\pm 1 \pm E_1 \pm E_2 \pm E_3\right)\right\} \tag{28}$$

are the quaternions forming the binary tedrahedral group and representing the roots of $SO(8)$. Once this set of octonions are given we can construct the root system of F_4 and form the roots of E_8 similar to the equation (5).

It is obvious that for a given $p(\bar{p})$ one can construct the elements of the binary tedrahedral group, in other words, $SO(8)$ root system 9 different ways as we have argued in the previous section. Since we have 112 roots of this type and a choice of p includes always $\pm p$ and $\pm\bar{p}$ that reduces such a choice to $\frac{112}{4} = 28$. This number further reduces to $\frac{28}{4} = 7$ because $\bar{p}, r_1, r_2, r_3$ come always in quartets. It is not only $p(\bar{p})$ rotates r_1, r_2, r_3 in the cyclic order but any one of them rotates the other three cyclically. One can show, for example, that

$$r_1\bar{p}\bar{r}_1 = r_2, r_1r_2\bar{r}_1 = r_3, r_1r_3\bar{r}_1 = \bar{p}. \tag{29}$$

The others satisfy similar relations. Therefore the choice of elements of a binary tedrahedral group or equivalently F_4 root system out of octonions is $9 \times 7 = 63$. Since the group preserving the quaternion structure is of order 192 the overall group which preserves the octonionic root system of E_7 is a group of order $192 \times 63 = 12096$. It has to be a subgroup of G_2 and the group is certainly the Chevalley group $G_2(2)$ [2].

MAXIMAL SUBGROUPS OF $G_2(2)$ AND THE MAXIMAL LIE ALGEBRAS OF E_7

There are four regular maximal Lie algebras of E_7 :

$E_6 \times U(1)$, $SU(2) \times SO(12)$, $SU(8)$, $SU(3) \times SU(6)$; and there are four maximal subgroups of the Chevalley group $G_2(2)$. It is interesting to see whether any relations between these groups and the octonionic root systems of these Lie algebras exist (See M. Koca and F. Karsch in reference [2]). There is a one-to-one correspondence between them but with one exception. When one imposes the invariance of the octonion algebra on the root system of $SU(3) \times SU(6)$ one obtains a group which is not maximal in the Chevalley group $G_2(2)$. Yet the maximal subgroup $[T, V_0] \oplus [T', V_1]$ of

order 192(17) preserves the quaternion algebra of the magic square structure (SP_3, F_4).The other maximal subgroups of $G_2(2)$ which are of orders $432(14), 192(14)$ and $336(9)$ have one-to-one correspondences with the groups which preserve the octonionic root systems of $E_6 \times U(1)$, $SU(2) \times SO(12)$ and $SU(8)$ respectively. In this section we will discuss the constructions of these three maximal subgroups of $G_2(2)$ as the automorphism groups of the corresponding octonionic root systems. Their character tables and the subgroup structures can be found in reference [27].

Octonionic root system of $E_6 \times U(1)$ and the group of order $432(14)$

Since $U(1)$ factor is represented by zero root we are essentially looking at the roots of E_6 in E_7 . Either using the simple roots of E_8 in Figure 1 or those roots of E_7 already given in equation (8) we may decompose the roots of E_7 to those roots orthogonal to the vector $\frac{1}{2}(1 - e_1 - e_2 - e_3)$ which constitute the 72 roots of E_6 and the ones having a scalar product $\pm\frac{1}{2}$ with it will be the weights of the representations $\underline{27} + \underline{27^*}$. In an explicit form they read:

Non-zero roots of E_6 :

$$\begin{aligned}
&\pm e_4, \pm e_5, \pm e_6, \frac{1}{2}(\pm e_4 \pm e_5 \pm e_6 \pm e_7), \pm\frac{1}{2}(e_2 - e_3 \pm e_4 \pm e_7), \pm\frac{1}{2}(e_2 - e_3 \pm e_5 \pm e_6),\\
&\pm\frac{1}{2}(e_3 - e_1 \pm e_6 \pm e_7), \pm\frac{1}{2}(e_3 - e_1 \pm e_4 \pm e_5),\\
&\pm\frac{1}{2}(e_1 - e_2 \pm e_5 \pm e_7), \pm\frac{1}{2}(e_1 - e_2 \pm e_4 \pm e_6)
\end{aligned} \tag{30}$$

The number in the bracket is the number of conjugacy classes and is used to distinguish the groups having the same order.

Weights of $\underline{27} + \underline{27^*}$ of E_6 :

$$\begin{aligned}
&\pm e_1, \pm e_2, \pm e_3\\
&\pm\frac{1}{2}(e_2 + e_3 \pm e_4 \pm e_7), \pm\frac{1}{2}(e_2 + e_3 \pm e_5 \pm e_6),\\
&\pm\frac{1}{2}(e_3 + e_1 \pm e_6 \pm e_7), \pm\frac{1}{2}(e_3 + e_1 \pm e_4 \pm e_5),\\
&\pm\frac{1}{2}(e_1 + e_2 \pm e_5 \pm e_7), \pm\frac{1}{2}(e_1 + e_2 \pm e_4 \pm e_6)
\end{aligned} \tag{31}$$

Now we are in a position to determine the subgroup of the group of order 192(17) which preserves this decomposition.

The magic square indicates that the root system of E_6 can be obtained by Cayley-Dickson procedure as the pair $(SU(3), F_4)$ which is clear from (30) where the roots of $(SU3)$ are represented by the short roots $\pm\frac{1}{2}(e_2 - e_3)$, $\pm\frac{1}{2}(e_3 - e_1)$, $\pm\frac{1}{2}(e_1 - e_2)$.

It can be shown that the subgroup of the group of order 192(17) preserving this system of roots where the imaginary unit e_7 is left invariant is the group generated by the elements,

$$[t, V_0], [\frac{1}{\sqrt{2}}(e_2 - e_3), V_1] \tag{32}$$

Here t is given by $t = \frac{1}{2}(1 + e_1 + e_2 + e_3)$. More explicitly we can write the elements of the group of interest as follows

$$[t, V_0] \subset [V_+, V_0], [\bar{t}, V_0] \subset [V_-, V_0], [1, V_0] \subset [V_0, V_0]; \tag{33}$$

$$\begin{aligned}
[\frac{1}{\sqrt{2}}(e_2 - e_3), V_1] &\subset [V_1, V_1], [\frac{1}{\sqrt{2}}(e_3 - e_1), V_1] \subset [V_2, V_1],\\
[\frac{1}{\sqrt{2}}(e_1 - e_2), V_1] &\subset [V_3, V_1].
\end{aligned} \tag{34}$$

Each set contains 8 elements hence the group is of order 48. We recall that in the decomposition of the root system of E_7 in (30) and (31) under E_6 the quaternions $\pm t(\pm\bar{t})$ and thereby the quaternionic imaginary units e_1, e_2, e_3 are used. This implies that the sum $\frac{1}{\sqrt{3}}(e_1 + e_2 + e_3)$ is left invariant under the transformations $tq\bar{t}$ where q is any octonion. This proves that the group of concern is a finite subgroup of $SU(3)$ acting in the 6-dimensional Euclidean subspace. The discussions through the relations (17-20) show that one can construct the root system of E_6 in (30), consequently those weights in (31), 9 different ways implying that the group of order preserving the root system of E_6 in (30) is a finite subgroup of $SU(3)$ of order $48 \times 9 = 432$ with 14 conjugacy classes. The 6×6 irreducible matrix representation of this group as well as its character table can be found in reference [27].

The octonionic root system of $SU(2)\times SO(12)$ and the group of order 192(14)

Existence of an automorpism group of order 192 is obvious since the $SU(2)$ roots are any imaginary octonion $\pm q$ which must be left invariant under any transformation. Since we have $126/2 = 63$ choice for the $SU(2)$ roots the group of invariance is $12096/63 = 192$. The structure of this group is totally different than the previous group of order 192(17) as we will discuss below.

The magic square tells us that the root system of $SO(12)$ can be obtained by pairing two sets of quaternionic roots of $SP(3)$ *a'la* Cayley-Dickson procedure $(SP(3), SP(3))$. When we take the quaternionic roots of $SP(3)$ given in(9) we obtain the root system of $SO(12)$ and $SU(2)$ as follows:

$SO(12)$ roots :

$$\begin{aligned}
\pm e_1, \pm e_2, \pm e_3, e_7(\pm e_1, \pm e_2, \pm e_3) &= \pm e_4, \pm e_5, \pm e_6 \\
\frac{1}{2}(\pm e_2 \pm e_3) + e_7\frac{1}{2}(\pm e_2 \pm e_3) &= \frac{1}{2}(\pm e_2 \pm e_3 \pm e_5 \pm e_6) \\
\frac{1}{2}(\pm e_3 \pm e_1) + e_7\frac{1}{2}(\pm e_1 \pm e_2) &= \frac{1}{2}(\pm e_1 \pm e_3 \pm e_4 \pm e_5) \\
\frac{1}{2}(\pm e_1 \pm e_2) + e_7\frac{1}{2}(\pm e_3 \pm e_1) &= \frac{1}{2}(\pm e_1 \pm e_2 \pm e_4 \pm e_6)
\end{aligned} \tag{35}$$

$SU(2)$ roots : $\pm e_7$.

The remaining roots transform as the weights of the representation $(\underline{2}, \underline{32}')$ under $SU(2)\times SO(12)$. Since the root $\pm e_7$ remains invariant under any transformation which preserves the decomposition of E_7 under $SU(2)\times SO(12)$ the group which we seek is a finite subgroup of $SU(3)$. We recall from the previous discussions that the quaternionic root system of $SP(3)$ is preserved by the octahedral group $[T,\bar{T}]\oplus[T\prime,T\prime]$. However, we seek a subgroup of $[T,V_0]\oplus[T\prime,V_1]$ which is also a subgroup of the octahedral group. Since we have V_0 and V_1 on the right of the pairs it should be $[\bar{V}_0, V_0]\oplus[\bar{V}_1, V_1]$. Actually we can write all the group elements explicitly,

$$[1,1], [e_1,-e_1], [\frac{1}{\sqrt{2}}(1+e_1), \frac{1}{\sqrt{2}}(1-e_1)], [\frac{1}{\sqrt{2}}(1-e_1), \frac{1}{\sqrt{2}}(1+e_1)] \tag{36}$$

$$[e_2,-e_2], [e_3,-e_3], [\frac{1}{\sqrt{2}}(e_2+e_3), -\frac{1}{\sqrt{2}}(e_2+e_3)], [\frac{1}{\sqrt{2}}(e_2-e_3), \frac{1}{\sqrt{2}}(-e_2+e_3)] \tag{37}$$

The elements in (37) form a cyclic group Z_4 and those in (37) are the right or left cosets of (37) with, say, $[e_2,-e_2]$ is a coset representative. Indeed the elements $[1,1]$ and $[e_2,-e_2]$ form the group Z_2 which leaves the group Z_4 invariant under conjugation. Hence the group of order 8 in (36-37) has the structure $Z_4 : Z_2$ where Z_4 is an invariant subgroup. We may also allow $e_7 \to -e_7$ that amounts to extending the group $Z_4 : Z_2$ by the element $[-1,1]$. Since the element $[-1,1]$ commutes with the elements of $Z_4 : Z_2$ then we have a group of order 16 with the structure $Z_2\times(Z_4 : Z_2)$. This is the group of automorphism of the root system in (35) when the quaternionic units are taken to be e_1, e_2 and e_3.

Now the question is how many different ways we decompose (35) allowing $e_7 \to \pm e_7$ only. In other words, what is the number of quaternionic units one can choose allowing $e_7 \to \pm e_7$. These units of quaternions can be chosen from the set of 112 roots orthogonal to e_7. They are

$$\frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3), \frac{1}{2}(\pm 1 \pm e_1 \pm e_5 \pm e_6), \frac{1}{2}(\pm 1 \pm e_2 \pm e_4 \pm e_5), \frac{1}{2}(\pm 1 \pm e_3 \pm e_4 \pm e_6). \tag{38}$$

One can prove that each set of 16 octonions in (38) will yield to 3 sets of quaternionic imaginary units not involving e_7. Therefore there are 12 different quaternionic units to build the group structure $Z_2x(Z_4 : Z_2)$ and the number of overall elements of the group preserving the root system in (35) is $12 \times 16 = 192$. To give a nontrivial example let us choose $p = \frac{1}{2}(1 + e_2 + e_4 + e_5)$ with $\bar{p} = \frac{1}{2}(1 - e_2 - e_4 - e_5)$. The following set of octonions chosen from(35)

$$\frac{1}{2}(\pm e_1 + e_2 + e_4 \pm e_6), \frac{1}{2}(\pm e_3 + e_2 + e_5 \pm e_6), \frac{1}{2}(\pm e_1 + e_4 + e_5 \pm e_3) \tag{39}$$

have scalar products $q_i.\bar{p} = 0$ where q_i is one of those in (39). Under the rotation $pq_i\bar{p}$, for example, the quaternionic units

$$E_1 = \frac{1}{2}(e_2 + e_5 + e_5 - e_6), E_2 = \frac{1}{2}(e_1 - e_3 + e_4 + e_5), E_3 = \frac{1}{2}(-e_1 + e_2 + e_4 + e_6) \tag{40}$$

are permuted and one can construct (35) with the set of octonions

$SO(12)$ roots:

$$\begin{aligned} \pm E_1, \pm E_2, \pm E_3, e_7(\pm E_1, \pm E_2, \pm E_3) &= \pm E_4, \pm E_5, \pm E_6 \\ \frac{1}{2}(\pm E_2 \pm E_3) + e_7\frac{1}{2}(\pm E_2 \pm E_3) &= \frac{1}{2}(\pm E_2 \pm E_3 \pm E_5 \pm E_6) \\ \frac{1}{2}(\pm E_3 \pm E_1) + e_7\frac{1}{2}(\pm E_1 \pm E_2) &= \frac{1}{2}(\pm E_1 \pm E_3 \pm E_4 \pm E_5) \\ \frac{1}{2}(\pm eE_1 \pm E_2) + e_7\frac{1}{2}(\pm E_3 \pm E_1) &= \frac{1}{2}(\pm E_1 \pm E_2 \pm E_4 \pm E_6) \end{aligned} \tag{41}$$

$SU(2)$ roots: $\pm e_7$

This is certainly invariant under the quaternion preserving automorphism group of order 16 as discussed above where the imaginary quaternionic units e_1, e_2, e_3 in (36-37) are replaced by E_1, E_2, E_3 in (40). One can proceed in the same manner and construct 12 different sets of quaternionic units by which one constructs the group $Z_2 \times (Z_4 : Z_2)$.

Octonionic root system of $SU(8)$ and the automophism group of order $336(9)$

Using the Coxeter-Dynkin diagram of figure1 we can write the octonionic roots of $SU(8)$ as follows:

$$\begin{aligned} &\pm e_1, \pm e_2, \pm e_4, \pm e_6 \\ &\frac{1}{2}(\pm e_1 \pm e_2 + e_5 + e_7), \frac{1}{2}(\pm e_1 \pm e_4 + e_3 + e_5), \frac{1}{2}(\pm e_1 \pm e_6 + e_3 + e_7) \\ &\frac{1}{2}(\pm e_2 \pm e_4 + e_3 - e_7), \frac{1}{2}(\pm e_2 \pm e_6 - e_3 + e_5), \frac{1}{2}(\pm e_4 \pm e_6 - e_5 + e_7). \end{aligned} \tag{42}$$

First of all, we note that the roots of E_7 decompose under its maximal Lie algebra $SU(8)$ as $126 = 56 + 70$. Therefore those roots of E_7 in (8) not displayed in (42) are the weights of the 70 dimensional representation of $SU(8)$.

To determine the automorphism group of the set in (42) we may follow the same method discussed above however here we choose a different way for $SU(8)$ is not in the magic square.

In an earlier paper [16] we have constructed the 7-dimensional irreducible representation of the group $PSL_2(7) : Z_2$ of order 336 and proved that this group preserves the octonionic root system of $SU(8)$. Below we give three matrix generators of the Klein's group $PSL_2(7)$, a simple group with 6 conjugacy classes,

$$A = \frac{1}{2}\begin{bmatrix} -1 & -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 1 \\ 1 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}; B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$C = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 & 0 \end{bmatrix} \tag{43}$$

These matrices satisfy the relation

$$A^4 = B^2 = C^7 = I. \tag{44}$$

The matrices A and B generate the octahedral subgroup of order 24 of the Klein's group.

The 56 octonionic roots can be decomposed into 7-sets of hyperocthedra in 4- dimensions. The matrix C permutes the seven sets of octahedra to each other. The octahedral group generated by A and B preserves one of the octahedra while transforming the other sets to each other. We display the 7-octahedra as follows:

$$\underline{1}:\ \begin{array}{c} \pm e_2 \\ \pm\frac{1}{2}(e_4-e_5+e_6+e_7) \\ \pm\frac{1}{2}(e_1-e_3+e_6-e_7) \\ \mp\frac{1}{2}(e_1+e_3+e_4+e_5) \end{array} \quad \underline{2}:\ \begin{array}{c} \pm e_1 \\ \pm\frac{1}{2}(e_2+e_3-e_5+e_6) \\ \pm\frac{1}{2}(-e_2+e_3-e_4-e_7) \\ \pm\frac{1}{2}(e_4+e_5+e_6-e_7) \end{array}$$

$$\underline{3}:\ \begin{array}{c} \mp e_4 \\ \pm\frac{1}{2}(e_1+e_3+e_6+e_7) \\ \pm\frac{1}{2}(-e_1-e_2+e_5+e_7) \\ \pm\frac{1}{2}(e_2-e_3+e_5+e_6) \end{array} \quad \underline{4}:\ \begin{array}{c} \pm\frac{1}{2}(-e_2-e_3+e_5+e_6) \\ \pm\frac{1}{2}(-e_4-e_5+e_6+e_7) \\ \pm\frac{1}{2}(-e_1-e_3+e_4-e_5) \\ \mp\frac{1}{2}(-e_1+e_2+e_5+e_7) \end{array}$$

$$\underline{5}:\ \begin{array}{c} \mp\frac{1}{2}(e_1+e_2+e_5+e_7) \\ \pm e_6 \\ \pm\frac{1}{2}(e_2+e_3+e_4-e_7) \\ \mp\frac{1}{2}(e_1-e_3+e_4-e_6) \end{array} \quad \underline{6}:\ \begin{array}{c} \pm\frac{1}{2}(-e_1+e_3+e_4+e_5) \\ \pm\frac{1}{2}(-e_2+e_3-e_5+e_6) \\ \pm\frac{1}{2}(e_1+e_3-e_6+e_7) \\ \mp\frac{1}{2}(e_2+e_3-e_4-e_7) \end{array}$$

$$\underline{7}:\ \begin{array}{c} \pm\frac{1}{2}(e_4-e_5-e_6+e_7) \\ \pm\frac{1}{2}(-e_1+e_3+e_6+e_7) \\ \pm\frac{1}{2}(e_2-e_3-e_4+e_7) \\ \mp\frac{1}{2}(e_1-e_2+e_5+e_7) \end{array}$$

Note that each vector is orthogonal to the others in a given set of 8 vectors forming an octahedron in 4-dimensions. The matrix C permutes the set of octahedra as $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 1$. The matrices A and B leave the set of vectors in $\underline{1}$ invariant and transforms the other sets to each others as follows:

$A : 2 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 2$; $3 \leftrightarrow 4$ and leaves 1 invariant.

$B : 3 \leftrightarrow 5$; $4 \leftrightarrow 7$ and leaves each of the set $1, 2, 6$ invariant.

When we decompose the weights of the 70 dimensional representation of $SU(8)$ under the octahedral group the vectors are partitioned as numbers of vectors $2, 6, 6, 8, 12, 12, 24$. The vector $\pm\frac{1}{2}(e_1+e_2+e_5-e_7)$ is left invariant under the octahedral group which corresponds to its 2 dimensional irreducible representation. The group $PSL_2(7)$ can be further extended to the group $PSL_2 : Z_2$ of order 336 by adding a generator which can be obtained from the transformation $e_1 \rightarrow -e_1, e_2 \rightarrow e_2, e_4 \rightarrow e_4$. One can readily check that the this transformation leaves the root system of $SU(8)$ invariant.

CONCLUSION

We have constructed the root system of E_8 from the quaternionic roots of F_4 *a'la* Cayley-Dickson doubling procedure that is a different realization of the magic square. The roots of E_7 are represented by the imaginary octonions which can be constructed by doubling the quaternionic roots of $SP(3)$ and F_4. The Weyl group of E_7 is isomorphic to the finite group $Z_2 \times SO_7(2)$ where $SO_7(2)$ is the adjoint Chevalley group over the finite field F_2. We have proven that the automorphism group of the octonionic root system of E_7 is the adjoint Chevalley group $G_2(2)$, a finite subgroup of the Lie group G_2 of order 12096. First we have determined one of its maximal subgroup of order 192 which preserves the quarternion subalgebra in the root system of E_7 and proven that this group can be embedded in the larger group 63 different ways. The other three maximal subgroups of orders $432, 192, 336$ respectively corresponding to the automorphism groups of the octonionic root systems of the maximal Lie algebras E_6, $SU(2) \times SO(12)$, $SU(8)$ have been studied in some depth. The root system of $SU(8)$ has a fascinating geometrical structure where the roots can be decomposed as 7 hyperoctahedra in 4-dimensions which are permuted to each other by one of the generators of the Klein's group $PSL_2(7)$.

Any one of these subgroups or the whole group $G_2(2)$ could be used to construct the manifolds which can be useful for the compactification of a theory in 11 dimension.

* Electronic address: kocam@squ.edu.om
† Electronic address: koc@gantep.edu.tr
‡ Electronic address: nazife@hotmail.com

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